CERTAIN PROBLEMS IN THE THEORY OF CRACKS IN A BEAM APPROXIMATION

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In the first section the problem concerning the motion of a crack under the action of loads applied to its edges is solved approximately in the same formulation as in [1]. In the second section we consider cracks whose propagation takes place because the bar loses its stability in longitudinal compression.

1. The propagation of a crack under load. Let a crack be located within the segment $0 \le x \le l$ of the center line of a semi-infinite ($x \ge 0$) bar of rectangular cross section height 2H, and width b, with the edges of the crack at $x = 0$ loaded by masses m located in a gravity field with acceleration g. We consider the part of the bar located on one side of the crack as a beam which is built-in at $x = l$. We denote the displacement of the neutral beam axis at point x at the instant t by u(x, t), the surface-energy density of the beam material by T, Young's modulus by E, the beam material density by ρ, and the linear density by ρbH. As shown in [1], the displacement u(x, t) and the length $l(t)$ of the crack must satisfy the relations

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad a^2 = \frac{EI}{\rho bH}, \tag{1.1}$$

$$u(l, t) = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0,$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=l} = A = \left(\frac{2Tb}{EI}\right)^{1/2}, \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0, \tag{1.2}$$

$$EI\left.\frac{\partial^3 u}{\partial x^3}\right|_{x=0} = mg - m\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=0}. \tag{1.3}$$

Using Eq. (1.1) we can transform condition (1.3) to the form

$$EI\left.\frac{\partial^3 u}{\partial x^3}\right|_{x=0} = mg + ma^2\left.\frac{\partial^4 u}{\partial x^4}\right|_{x=0}. \tag{1.4}$$

We use the Kantorovich method to reduce problem (1.1), (1.2), (1.4) to a system of ordinary differential equations [2]. Approximately we represent the displacement u(x, t) in the form of a polynomial in degrees of $(x - l)$, with its coefficients chosen such that boundary conditions (1.2), (1.4) would be satisfied for any l. Then $l(t)$ and the coefficients that were left undetermined are determined from the system of Euler equations which must be satisfied for a minimum of the integral

$$S = \int_{t_1}^{t_2}\int_0^{l(t)}\left[\frac{\rho bH}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{EI}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right]dx\,dt +$$
$$+ \int_{t_1}^{t_2}\left[\frac{m}{2}\left(\frac{\partial u(0, t)}{\partial t}\right)^2 + mgu(0, t) - Tbl(t)\right]dt. \tag{1.5}$$

We confine ourselves to the case of one equation, i.e., we approximate the displacement with the fourth degree polynomial

$$u(x, t) \approx {}^1/_2\, A(x - l)^2 + {}^1/_6\, Al^{-1}[1 - \gamma(L - 1)] \times$$
$$\times (x - l)^3 - {}^1/_{12}\, Al^{-2}\gamma(L - 1)(x - l)^4,$$
$$L = \frac{l}{l_0}, \quad l_0 = \frac{AEI}{mg}, \quad \gamma = \frac{LM}{2 + LM},$$
$$M = \frac{\rho bHl_0}{m} = \frac{\sqrt{6}\,\rho g}{\sqrt{ET}}\left(\frac{l_0}{H}\right)^2 H^{3/2}. \tag{1.6}$$

Substituting (1.6) into (1.5), we obtain

$$S = \frac{Aml_0^2}{6}\int_{t_1}^{t_2}\left[Al_0^2\Phi(L)\left(\frac{dL}{dt}\right)^2 + g\varphi(L)\right]dt,$$

$$\Phi(L) = L^3 M\left[\frac{61}{420} + \frac{39\gamma}{1120} + \frac{23\gamma^2}{6048} +\right.$$
$$+ \gamma(L - 1)\left(\frac{37}{210} - \frac{\gamma}{1680} - \frac{23\gamma^2}{3024}\right) +$$
$$\left.+ \gamma^2(L - 1)^2\left(\frac{11}{140} - \frac{23\gamma}{672} + \frac{23\gamma^2}{6048}\right)\right] +$$
$$+ L^2\left[\frac{4}{3} + \frac{\gamma}{3} + \frac{\gamma^2}{48} + \gamma(L - 1)\left(\frac{4}{3} - \frac{\gamma}{2} + \frac{\gamma^2}{24}\right) +\right.$$
$$\left.+ \gamma^2(L - 1)^2\left(\frac{1}{3} - \frac{\gamma}{6} + \frac{\gamma^2}{48}\right)\right],$$
$$\varphi(L) = L\left[-4 + 2L + \gamma^2(L - 1)^2\frac{5 + 2LM}{5LM}\right].$$

We set up the Euler equation for determining L(t); we obtain

$$2Al_0^2\Phi(L)\frac{d^2L}{dt^2} + Al_0^2\frac{d\Phi(L)}{dL}\left(\frac{dL}{dt}\right)^2 = g\frac{d\varphi(L)}{dL}\quad\frac{dl}{dt}.$$

By the substitution $y(L) = (dL/dt)^2$ this equation is reduced to a first order linear equation. The solution of this equation gives the velocity of the crack as a function of

$$\frac{dL}{dt} = \left[\frac{6B/Aml_0^2 + g\varphi(L)}{Al_0^2\Phi(L)}\right]^{1/2}. \tag{1.7}$$

The integration constant B is the total energy of the system.

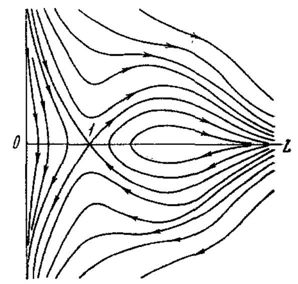

Fig. 1

The equilibrium position is determined from the conditions $dL/dt = 0$, $d^2L/dt^2 = 0$, i.e., as a root of the equation $d\varphi/dL = 0$. Solving this equation, we obtain $L = 1$. Consequently, l_0 from (1.6) is the equilibrium length of the crack. The form of the phase plane (L, dL/dt) is shown in Fig. 1. From this we see that the position of equilibrium is unstable. The initial configuration of displacements and velocities in the given approximation is connected with formulas (1.6) and (1.7).

If we cannot assume that the initial state satisfied this condition, then the approximation just considered is inadequate, and the displacement must be approximated by a polynomial of a higher degree. Here new constants of integration appear which enable us to extend the set of admissible initial states.

Let us consider in more detail the phase trajectory passing through the position of equilibrium ($B = 1/3)\, A^2EIl_0$). We introduce the dimensionless velocity $\theta = (l/a)(dl/dt)$, equal to the ratio of the crack velocity to the velocity of a flexure wave whose length equals the length of the crack. Then Eq. (1.7) is rewritten as follows:

$$\theta^2 = M\left(\frac{L - 1}{L}\right)^2\left[2 + \frac{L^2M(5 + 2LM)}{5(2 + LM)^2}\right]\Phi(L). \tag{1.8}$$

Relation (1.8) is presented in Fig. 2 in (L, θ) coordinates for various values of the parameter M for a crack moving from left to right. $M = 0$ formulas (1.6), (1.7) give the exact solution of the problem formulated here. This case corresponds to the propagation of the crack in an inertia-free bar. Formula (1.7) in this case assumes the form

$$\frac{dL}{dt} = \frac{1}{l_0}\left(\frac{3g}{2A}\right)^{1/2}\left(\frac{L-1}{L}\right).$$

Consequently, as the crack grows from the position of equilibrium, its velocity tends to the value $(3g/2A)^{1/2}$. The case $M = \infty$ corresponds to the propagation of the crack under a constant force mg applied at the end $x = 0$. This case is obtained from the problem considered here by going to the limit $m \to 0$, $g \to \infty$, $mg = \text{const}$.

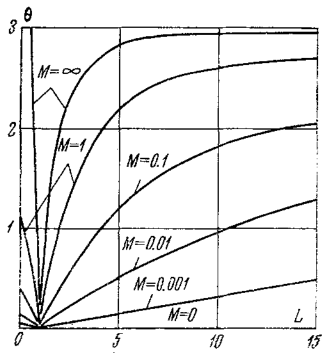

Fig. 2

In contrast to the case $M = 0$, here the inertia of the load material is taken into account, but the inertia of the load is not taken into consideration.

As we see from expression (1.7) for $\Phi(L)$, however small the parameter $M \neq 0$, for sufficiently large L the motion takes place as if $M = \infty$. Thus, the quasi-static approximation $M = 0$ is applicable for $L \ll (4/3)(420/61) M^{-1}$.

We present numerical values of M for $H = 0.5$ cm, $l_0/H = 20$, $g = 981$ cm/sec^2 (the last formula in (1.6) is used). For α-Fe: $\rho = 7.86$ g/cm^3, $T = 1450$ dyne/cm, $E = 13.2 \cdot 10^{11}$ dyne/cm^2, $M = 0.06$; for LiF: $\rho = 2.29$ g/cm^3, $T = 700$ dyne/cm, $E = 7.35 \cdot 10^{11}$ dyne/cm^2, $M = 0.03$. The fairly typical values of M presented here show that hardly ever is the case $M = \infty$ encountered in experiments on cracks.

2. Cracks under the conditions of longitudinal flexure. Let a bar of infinite length $-\infty < x < \infty$, whose dimensions and properties are the same as in Section 1, be compressed by a longitudinal force whose absolute value is denoted by P. We assume that as a result of a loss of stability, a crack located within the segment $-l \le x \le l$ develops symmetrically along the center line of the bar. We assume that the loss of stability takes place only over that segment in which the crack is located, with the ends of the crack fixed rigidly. We neglect the longitudinal compressibility of the bar and assume that the force P does work only over the displacement associated with the buckling of the bar. In approximation of slight flexure this displacement equals [3]

$$\int_{-l}^{l} \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

The part of the bar located outside the segment $-l \le x \le l$, is called the rigid part of the bar. Using the principle of least work, we can show, in analogy with what was done in [1] for the case of transverse flexure, that the displacement u(x, t) of the neutral axis of the bar having lost stability satisfies the equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{P}{EI}\frac{\partial^2 u}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad a^2 = \frac{EI}{\rho b H} \tag{2.1}$$

and the boundary conditions

$$u = 0, \quad \partial u / \partial x = 0, \quad \partial^2 u / \partial x^2 = A = (2Tb / EI)^{1/2} \tag{2.2}$$

at each end of the crack. Let us consider certain cases of crack equilibrium and propagation. Owing to the symmetry of the cases considered below, the solution is found for $0 \le x \le l$; for $x = 0$ the conditions

$$\partial u / \partial x = 0, \qquad \partial^3 u / \partial x^3 = 0, \tag{2.3}$$

are satisfied.

Furthermore, it is natural to require that $u = 0$ only for $x = l$; this eliminates contact between the opposing edges of the crack.

a) Static problems. If displacement is not a function of time, the inertia term vanishes in Eq. (2.1). Let the force P be given. Then

$$u(x) = AEIP^{-1}(1 + \cos\sqrt{P/EI}\,x), \qquad l = \pi\sqrt{EIP^{-1}}. \tag{2.4}$$

Since a smaller force corresponds to a larger equilibrium length of the crack, the equilibrium considered here is unstable. More natural is the formulation of the problem, such that the displacement s of the rigid part of the bar is specified (relative to the point $x = 0$). Such a case can be realized, for example, with slow shifting of the testing machine clamps. The displacement and length of the crack in this case are expressed by the formulas (2.4), but the force P is now unknown. With the second formula in (2.4) the displacement should be expressed in terms of the length of the crack, while the length itself must be determined from the relation

$$s = \int_0^l \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Carrying out the necessary calculations, we obtain

$$l = \frac{2^{2/3}\pi^{2/3}s^{1/3}}{A^{2/3}}, \qquad P = \frac{\pi^2 EIA^{4/3}}{2^{4/3}s^{2/3}}. \tag{2.5}$$

Equilibrium in this case will be stable. An infinite increase in P as $s \to 0$ is caused by assumption of the incompressibility of the bar. Apparently it is advisable to take into account the longitudinal compressibility for small s, since for small s the length of the crack is also small and the simple theory of longitudinal flexure used here becomes inapplicable.

b) Dynamic problems. Analogous to [1] we can obtain an invariant-group solution of the problem concerning the propagation of a crack with consideration of the inertia term in (2.1). Such solutions are stationary solutions of the form $u(x, t) = u(x - Vt)$ and self-similar solutions of the form $u(x, t) = Utf(x^2/at)$ (here V and U are constants having the dimensions of velocity; f is a dimensionless function). Solutions of the first type describe the propagation of a crack of constant length along the bar. In view of the irreversibility of real cracks, this is of no interest. Therefore they are not considered here.

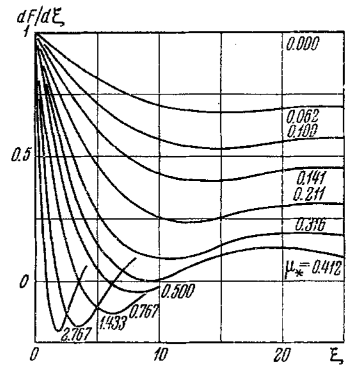

Fig. 3

Let us consider solutions of the second type. They describe the propagation of a crack in such a way that the configurations of the system at any two instants of time t_1 and t_2 can be transformed into one another by stretching them along the axis of displacements (t_2/t_1) times and along the x axis $(t_2/t_1)^{1/2}$ times. It is obvious that if the boundary conditions (2.2) and (2.3) are satisfied for $t = t_1$, they will be satisfied also for $t = t_2$. If such propagation is possible, the crack must develop according to the law $l^2 = \lambda$, where λ is a dimensionless quantity depending on the conditions of the experiment, but not varying in the process of crack development. From dimensional analysis it follows that the longitudinal force must vary in inverse proportion to the square of the crack length, i.e., the equation $P = C/l^2$ must be satisfied, where C is a constant, dimensional quantity. We introduce the notations $\xi = x^2/at$, $\mu = C/4EI\lambda$ and make the substitution $u(x, t) = Utf(\xi)$ in Eq. (2.1) and the boundary conditions (2.2) and (2.3). We obtain the following problem:

$$\xi^2 \frac{d^4 f}{d\xi^4} + 3\xi \frac{d^3 f}{d\xi^3} + \left(\frac{3}{4} + \mu\xi + \frac{\xi^2}{16}\right) \frac{d^2 f}{d\xi^2} + \frac{\mu}{2} \frac{df}{d\xi} = 0 ,$$

$$\lim_{\xi \to 0} \sqrt{\xi} \frac{df}{d\xi} = 0, \qquad \lim_{\xi \to 0} \sqrt{\xi} \left(3 \frac{d^2 f}{d\xi^2} + 2\xi \frac{d^3 f}{d\xi^3}\right) = 0 ,$$

$$f(\lambda) = 0, \qquad \frac{df(\lambda)}{d\xi} = 0, \qquad 4\lambda \frac{d^2 f(\lambda)}{d\xi^2} = \frac{Aa}{U}. \tag{2.6}$$

We find the solution of this problem in the form of the power series in ξ

$$f(\xi) = aAU^{-1} [4\lambda d^2 F(\lambda) / d\xi^2]^{-1} [F(\xi) - F(\lambda)] .$$

Here $F(\xi)$ is some power series, which is not written out; λ is the first root of the equation $dF/d\xi = 0$, lying on the positive part of the semiaxis, with $dF/d\xi$ determined by the relations

$$\frac{dF}{d\xi} = \sum_{k=0}^{\infty} \beta_k, \qquad \beta_0 = 1, \qquad \beta_1 = -\frac{2}{3}\mu\xi,$$

$$\beta_{k+1} = -\frac{\xi}{(k+1)(k+{}^3/_2)} \left[\mu\beta_k + \frac{(k-1)\xi}{16(k+{}^1/_2)} \beta_{k-1}\right]. \tag{2.7}$$

Using the recursive relation for the terms of the series (2.7), we can easily show that $\lim(\beta_{k+1}/\beta_k) = 0$ as $k \to \infty$ for all ξ and μ, and hence, according to the d'Alembert criterion, the series determining $dF/d\xi$ converges everywhere. The existence of a solution for the problem (2.6) thus depends on whether there exist positive zeros for series (2.7). For $\xi \ll \mu$, discarding the second term in the recursive formula (2.7), we find that

$$dF / d\xi \approx \sin \sqrt{4\mu\xi} / \sqrt{4\mu\xi} .$$

Therefore, if μ is sufficiently large, $\lambda \approx \pi^2/4\mu$. We now prove that if μ is sufficiently small, series (2.7) has no positive zeros. For this we introduce the functions $v(\xi)$, $w(\xi)$ such that

$$\frac{d^2 F}{d\xi^2} = \mu v(\xi) = \xi^{-3/2} w(\xi), \qquad \frac{dF}{d\xi} = 1 + \mu \int_0^{\xi} v(\tau)\, d\tau .$$

The problem consequently consists of proving the boundedness of

$$\int_0^{\xi} v(\tau)\, d\tau .$$

We introduce the functions $v(\xi)$, $w(\xi)$ into the differential equation (2.6)

$$\frac{d^2 w}{d\xi^2} + \frac{w}{16} = -\mu \left[\frac{w}{\xi} + \frac{1}{2\sqrt{\xi}} \left(1 + \mu \int_0^{\xi} v(\tau)\, d\tau\right)\right].$$

Using the formula for solving a nonhomogeneous linear equation with constant coefficients [4], we obtain the integral equation for $v(\xi)$ (the arbitrary constants in the general solution of the homogenous solution are put equal to zero for the behavior of $v(\xi)$ in the neighborhood of a zero to agree with (2.7)):

$$\xi^{3/2} v(\xi) = -4\mu \int_0^{\xi} \sqrt{\eta}\, v(\eta) \sin \frac{\xi - \eta}{4}\, d\eta -$$

$$-4\mu \int_0^{\xi} \frac{\sin {}^1/_4(\xi - \eta)}{2\sqrt{\eta}} \int_0^{\eta} v(\tau) d\tau + 4 \int_0^{\xi} \frac{\sin {}^1/_4(\xi - \eta)}{2\sqrt{\eta}}\, d\eta . \tag{2.8}$$

We denote by $R(\xi)$ the maximum value of $|v(\eta)|$ for $0 \le \eta \le \xi$. We evaluate the right side of (2.8)

$$\xi^{3/2} |v(\xi)| \leqslant 4\mu \frac{3}{2} R(\xi) \left| \int_0^{\xi} \sqrt{\eta} \sin \frac{\xi - \eta}{4}\, d\eta \right| +$$

$$+ 4 \left| \int_0^{\xi} \frac{\sin [{}^1/_4(\xi - \eta)]}{2\sqrt{\eta}}\, d\eta \right| \leqslant$$

$$\leqslant 6\mu R(\xi)\, 4\sqrt{\xi} + 24\mu R(\xi) \left| \int_0^{\xi} \frac{\cos {}^1/_4(\xi - \eta)}{2\sqrt{\eta}}\, d\eta \right| +$$

$$+ 4 \left| \int_0^{\xi} \frac{\sin {}^1/_4(\xi - \eta)}{2\sqrt{\eta}}\, d\eta \right| \leqslant 24\mu R(\xi) \sqrt{\xi} +$$

$$+ 24\mu R(\xi)\, Q_1 + 4Q_1 .$$

Here Q_1 is a constant limiting in absolute value both integrals in the last inequality. It is now clear that $v(\xi)$ is bounded, because

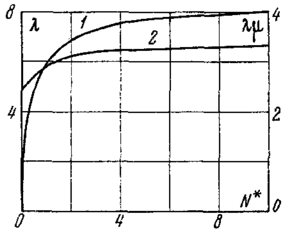

Fig. 4

otherwise we could find such values of ξ, larger than any previously given positive number, that $|v(\xi)| = R(\xi)$ ($v(\xi)$, being an entire function, is bounded at all finite points) and the inequality just proved would assume the form

$$R(\xi) \leqslant 24 \frac{\mu R(\xi)}{\xi} + \frac{24\mu R(\xi)\, Q_1 + 4Q_1}{\xi^{3/2}} ,$$

which for large ξ is contradictory. Consequently, for sufficiently large ξ the inequality $|v| < 48\mu R(\infty)/\xi + 4Q_1/\xi^{3/2}$ is valid. Using this inequality and Eq. (2.8), we find that for sufficiently large ξ we have $|v(\xi)| < Q_2/\xi^{3/2}$, where Q_2 is some constant. Hence follows the required boundedness of the integral

$$\int_0^{\xi} v(\tau)\, d\tau .$$

Figure 3 shows the graphs of the function $dF/d\xi$ for various values of μ, obtained by direct calculation for formulas (2.7). As is seen from these graphs, λ exists only for $\mu \ge \mu_* \approx 0.412$. Calculating the integral

$$s = \int_0^{l} \frac{1}{2} \left(\frac{\partial u}{\partial x}\right)^2 dx ,$$

we find that when solutions exist, the rigid part of the bar is displaced according to the law $s = Nt^{3/2}$ (N is a dimensional constant), and

$$\frac{N}{A^2 a^{3/2}} = \left[4\lambda \frac{d^2 F(\lambda)}{d\xi^2}\right]^{-2} \int_0^{\lambda} \sqrt{\xi} \left(\frac{dF}{d\xi}\right)^2 d\xi . \tag{2.9}$$

It is natural to assume that the quantity N is specified; then λ is determined from (2.9). In the case of large μ, i.e., small λ, we have

$$\frac{N}{A^2 a^{3/2}} = \frac{\lambda^{3/2}}{4\pi^2} , \qquad P = \frac{C}{l^2} = \frac{4EI\lambda\mu}{l^2} = \frac{EI\pi^2}{l^2} .$$

This case corresponds to quasi-static development of a crack; this can easily be shown by comparing the relations thus obtained with formulas (2.5). Passing from large to small μ and observing the behavior of λ, we necessarily arrive at a curve $\mu = \mu_*$ such that $\lambda = \lambda_*$ is a zero of the second order. This follows from $dF/d\xi$ as a continuous function of μ. Therefore $d^2F(\lambda)/d\xi^2$ tends to zero, as $\lambda \to \lambda_*$ while $N/A^2a^{3/2}$ increases without bounds. The graph

$\lambda(N/A^2 a^{3/2})$ (Fig. 4, curve 1) asymptotically approaches $\lambda = \lambda_* \approx 9.1$. From the same graph it is seen that the static solution is applicable to $N/A^2a^{3/2} \approx 0.1$. The absence of other branches for the curve $\lambda(N/A^2a^{3/2})$ follows from the fact that for $\mu < \mu_*$ functions (2.7) have no positive zeros. The last statement, although not proved, is very likely, as is seen from Fig. 3.

Thus, if the displacement of the rigid part of the bar is realized according to the law $s = Nt^{3/2}$ and $l(0) = 0$, then

$$u(x, t) = aAt\left[4\lambda \frac{d^2F(\lambda)}{d\xi}\right]^{-1}[F(\xi) - F(\lambda)],$$

$$l(t) = \left[\lambda\left(\frac{N}{A^2a^{3/2}}\right) at\right]^{1/2},$$

where the function $F(\xi)$ is determined from (2.7) with accuracy up to an additive constant, $\lambda(N/A^2a^{3/2})$ is given by curve 1 in Fig. 3, the longitudinal force is $P = 4EI\lambda\mu/l^2$, while the curve $\lambda\mu$ is given in Fig. 4 (curve 2).

In conclusion the author thanks Yu. E. Zhumakhev for help in carrying out the calculations on an M-20 computer.

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